About the slowing down of accelerated clocks revisited

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Abstract

The convergence of the integral that gives the proper time as a function of inertial time, when speed tends to the speed of light fast enough as inertial time tends to infinity, was studied by Suárez and Ferrari in 1985.Here this problem is reconsidered, in the framework of special relativity. Nothing special (besides the rate of growth towards infinity) seems to characterize the behavior of the tangent component of the fields of 3-force along the path of the accelerated clock, in relation with the convergence or divergence of the integral that gives proper time as function of inertial time. However, seen from the viewpoint of the proper acceleration of the clock, a physically meaningful difference appears between the tangential proper acceleration histories that give a finite proper time for an infinite inertial time, and those tangential proper 3-acceleration histories that give an infinite proper time for an infinite inertial time: either a singularity in proper 3-acceleration for a finite value of proper time in one case, or its absence in the other case.

(A) Introduction

Let us consider an inertial observer that follows the movement of an accelerated clock.

Let us agree, as usual, that a clock is any physical device that can be considered as a particle, follows a time-like world-line and generates a sequence of events called ticks. An ideal clock is a clock such that the proper time elapsed between any two (non-necessarily consecutive) ticks is proportional to the number of ticks between the first tick and the last tick considered, with the same proportionality factor at each point of the clock's world-line.

This definition presupposes the definition of proper time elapsed between two events along a time-like trajectory in Minkowski space-time, as the length of the segment of world-line between these two events, determined from the metric tensor in that space-time (Gourgoulhon, 2013).

So, an ideal clock is a clock that measures proper time. If a world-line of events in Minkowski space is described relative to an inertial system, then the proper time along this world-line can be calculated by means of a formula relating proper time to inertial time.

According to the *clock hypothesis* if an ideal clock moves non-uniformly through an inertial frame, acceleration as such has no effect on the rate of the clock: its instantaneous rate depends only on its instantaneous speed. The instantaneous rate of the accelerated clock is assumed to be identical to the rate of a clock fixed to a momentarily commoving inertial frame (Moller, 1952; Rindler, 1982; Brown, 2005).

In special relativity, if we accept the clock hypothesis, proper time is given as a function of inertial time, for all accelerated observers, by the known formula:

$$\tau(t) = \tau(t_0) + \int_{t_0}^{t} \sqrt{1 - \frac{v^2(t')}{c^2}}\, dt' \qquad (1)$$

Here, as usual, c is the speed of light in vacuum and t_0 is an initial value of inertial time. Now, as the rest mass of a clock is assumed non zero, its speed history $v(t)$ is constrained by the strict inequality $v(t) < c$ for every instant of inertial time.

An interesting possibility was conceived by José Ferrari in 1984: the convergence of the integral that gives the proper time as a function of inertial time, when $v(t)$ tends to c fast enough as t tends to infinity.

To explore possible physical consequences of the convergence of $\int_{t_0}^{+\infty} \sqrt{1 - \frac{v^2(t)}{c^2}}\, dt$

let us define:

$$\tau_s = \lim_{t \to +\infty} \tau(t) = \tau(t_0) + \int_{t_0}^{+\infty} \sqrt{1 - \frac{v^2(t)}{c^2}}\, dt \qquad (2)$$

Then it wouldn't be possible to assign a value of inertial time t to a proper time $\tau > \tau_s$.

To analyze the implications of this impossibility derived from the convergence of the integral that gives proper time as a function of inertial time, let us remember that the theory of special relativity stresses certain facts on which *every observer in the universe* (inertial or non-inertial) should agree, allowing for the unity of space-time. Events like explosions and implosions, births and deaths. Events that that take place in the neighborhood of a given clock and their temporal order. Conservation laws (momentum, energy). (Taylor and Wheeler, 1992)

A necessary condition to attain the abovementioned agreement about events is the possibility to establish a one to one correspondence between each instant of time t assigned to the moving clock in the inertial frame S and an instant of proper time τ corresponding to the same event, measured with the moving clock.

From (1) it follows: $\frac{d}{dt}\tau(t) = \sqrt{1 - \frac{v^2(t)}{c^2}} > 0$ The correspondence between inertial time and proper time given by equation (1) is strictly increasing and there is a one-to-one correspondence between inertial times and the proper times of the accelerated clock.

Up to this point, all seems right. But now, let us admit (as is done in almost all branches of physics, except physical cosmology and perhaps, astrophysics) that *time is unbounded*.

Then, if the integral $\int_{t_0}^{+\infty} \sqrt{1 - \frac{v^2(t)}{c^2}}\, dt$ happens to be convergent and proper time is not bounded, *part of the evolution of the accelerated observer should occur out of the universe of an inertial observer*.

Then births and deaths, explosions and implosions that could be part of the experience of the accelerated observer after τ_s couldn't be part of the experience of the inertial

observer. So, it seems that we could have a problem here, very different indeed to the "twin paradox". [1]

This problem motivated a preliminary discussion in a letter to Il Nuovo Cimento published by Suárez and Ferrari in 1985. Two alternative points of view were contemplated in the above mentioned letter to solve the problem: either the clock hypothesis is only an approximation valid under certain restrictions, or if it is always applicable, then we can only find in nature fields of force relative to the inertial observer such that $\int_{t_0}^{+\infty} \sqrt{1-\frac{v^2(t)}{c^2}}\ dt$ is always divergent (Suárez and Ferrari, 1985).

The purposes of the present work are:

(a) To study more in depth, relative to an inertial observer, the relationship between the relativistic force acting on the accelerating clock and the behavior of the integral that gives the proper time of the clock as function of inertial time. This will be done in next section B.

(b) To relate the convergence or divergence of the proper time integral with the behavior of the proper acceleration of the clock. This problem will be studied in section C.

(c) To study three examples of speed histories applying the results obtained in sections B and C. This will be done in section D.

(d) To discuss the meaning of the obtained results in a wider framework. The discussion and some conclusions will be presented in section E.

The analysis of the movement of the accelerated clock will be done in the framework of special relativity. [2] The accelerating clock can be dealt with in an inertial system:

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[1] Consider two ideal clocks. A first clock, fixed to an inertial system. The inertial time, given by this clock is represented by t . A second clock, that until the instant t_i is at rest next to the first clock and synchronized with it. From the instant t_i onwards the second clock departs from the first and travels following a trajectory with velocity $\vec{v}(t)$ with respect to the inertial system. At a later time t_f the second clock finds the first one again, and stops next to it. Being a closed trajectory, the possible histories of velocity verify the restriction: $\int_{t_i}^{t_f} \vec{v}(t)\, dt = \vec{0}$ But they are, otherwise, arbitrary. The elapsed inertial time, measured by the fixed clock is: $t_f - t_i$ The elapsed time *measured by the accelerated clock* (being $v(t)$ the speed history that corresponds to the velocity history $\vec{v}(t)$), *according to the clock hypothesis* is: $\int_{t_i}^{t_f} \sqrt{1-\frac{v^2(t)}{c^2}}\ dt$ This time interval is always less than the time interval $(t_f - t_i)$ measured by the inertial clock. *This is the twin paradox* if proper time is assumed to be also the most convenient time to measure of biological processes. The possible convergence of $\int_{t_i}^{+\infty} \sqrt{1-\frac{v^2(t)}{c^2}}\ dt$ is another thing, although of course it presupposes time dilation of accelerated clocks.

[2] The boundary between special and generalized relativity is usually established based on the curvature of space-time, not in a distinction between un-accelerated and accelerated observers (Hobson et al, 2006; Rindler, 2006; Gourgoulhon, 2013).

accelerating frames of reference are not needed in this case, although they can be used in special relativity. [3]

(B) The clock hypothesis and the behavior of fields of force relative to the inertial observer

Let us describe the movement of an accelerated clock (considered as a massive particle of rest mass m_0) relative to an inertial frame S by its position vector $\vec{r}(t)$, a regular function of the inertial time for $t \geq t_0$. The path followed by the clock is represented by Γ . The arc of path between an initial point P_0 of coordinates $(t_0, \vec{r}(t_0))$ and a variable final point P of coordinates $\left(t, \vec{r}(t)\right)$ is represented by $\Gamma_{P_0 P}$.

Because $\frac{d\vec{r}(t)}{dt} = \vec{v}(t)$ and its norm $\left\|\frac{d\vec{r}(t)}{dt}\right\|$ is the speed $v(t)$ of the clock, the length $l_{\Gamma_{P_0 P}}$ of this arc is given by:

$$l(t) = l_{\Gamma_{P_0 P}} = \int_{t_0}^{t} \left\|\frac{d\vec{r}(t')}{dt}\right\| \, dt' = \int_{t_0}^{t} v(t')\, dt' \qquad (3)$$

From (3) it follows that $\frac{dl(t)}{dt} = v(t)$. Assuming that $v(t) > 0$ for $t \geq t_0$ the derivative of arc length with respect to inertial time is always positive, so $l(t)$ is a strictly increasing function of inertial time. Furthermore, we suppose that $\lim_{t \to +\infty} l(t) = +\infty$ Then inverse function $t(l)$ exists for every $l \geq l_0$

As consequence, to each value of the inertial time $t \geq t_0$ it corresponds one value of arc length $l \geq l_0$, and vice versa[4].

Let us introduce the unit tangent vector at each point of the path:

$$\vec{t}_\Gamma(t) = \vec{t}_\Gamma\left(t(l)\right) = \frac{d\vec{r}\left(t(l)\right)}{dl} \qquad (4)$$

Then the clock velocity $\vec{v}$ and linear momentum $\vec{p}$ are given by:

$$\vec{v}(t) = v(t)\, \vec{t}_\Gamma(t) \qquad (5) \qquad\qquad \vec{p}(t) = p(t)\, \vec{t}_\Gamma(t) \qquad (6)$$

If m_0 is the rest mass of the clock and $\gamma_v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$ is its gamma factor,

$$p(t) = m_0\, \gamma_v\, v(t) \qquad (7)$$

[3] In special relativity, unlike what happens in generalized relativity, a distinction is made between inertial and non-inertial reference systems. However, in the restricted theory, physical equations in accelerating reference frames are described in flat Minkowski space-time introducing a metric tensor, using formal tools quite similar to that used in generalized relativity (Hobson et al, 2006; Rindler, 2006; Gourgoulhon, 2013). In an inertial frame the description of the clock movement is much simpler.

[4] The path of the clock can cross the same points of 3-D Euclidian space more than once. For example, this happens during a circular motion.

From equation (6) it follows that the relativistic extension of Newton's second law can be written, introducing a relativistic 3-force $\vec{f}$:

$$\frac{dp(t)}{dt}\,\vec{t}_\Gamma(t) + p(t)\,\frac{d\vec{t}_\Gamma(t)}{dt} = \vec{f}(t) \tag{8}$$

In what follows *we suppose that the relativistic 3-force is a smooth function, either of the inertial time or the position on the path* Γ followed by the accelerated clock relative to the inertial frame S .

Let us represent by $\cdot$ the scalar product in 3-D Euclidian space. Because $\left\|\vec{t}_\Gamma(t)\right\| = 1$ for every time instant, $\vec{t}_\Gamma(t)\cdot\frac{d\vec{t}_\Gamma(t)}{dt} = 0$ From this last equality and from (8) we obtain:

$$\frac{dp}{dt} = \vec{f}\cdot\vec{t}_\Gamma \tag{9}$$

From (9) it follows:

$$p(t) = p(t_0) + \int_{t_0}^{t} \vec{f}(t')\;\cdot\vec{t}_\Gamma(t')\,dt' \tag{10}$$

But $1-\frac{v^2}{c^2} = \frac{1}{1+\frac{p^2}{{m_0}^2\,c^2}}$ so from this last equality and from (10), we derive:

$$\int_{t_0}^{+\infty}\sqrt{1-\frac{v^2(t)}{c^2}}\,dt = \int_{t_0}^{+\infty}\frac{1}{\sqrt{1+\frac{\left(p(t_0)+\int_{t_0}^{t}\vec{f}(t')\cdot\vec{t}_\Gamma(t')\cdot dt'\right)^2}{{m_0}^2\,c^2}}}\,dt \tag{11}$$

If the relativistic 3-force is given as a function of position it is possible to make a change of integration variable in the integral that gives the proper time (taken on the path Γ of the accelerated clock), from inertial time to arc length.

Taking into account the equalities $v = \frac{\frac{p}{m_0}}{\sqrt{1+\frac{p^2}{{m_0}^2\,c^2}}}$ and $\sqrt{1-\frac{v^2}{c^2}} = \frac{1}{\sqrt{1+\frac{p^2}{{m_0}^2\,c^2}}}$:

$$\int_{t_0}^{+\infty}\sqrt{1-\frac{v^2(t)}{c^2}}\,dt = \int_{l_0}^{+\infty}\sqrt{1-\frac{v^2\big(t(l)\big)}{c^2}}\,\frac{1}{v\big(t(l)\big)}\,dl \;= m_0\int_{l_0}^{+\infty}\frac{1}{p\big(t(l)\big)}\,dl \tag{12}$$

On the 3-D path of the accelerated clock, there is a one to one correspondence between $\vec{r}(t)$ and $l(t)$, so the relativistic 3-force can be written as a function $\vec{f}(l)$ of the arc length l. From $v = \frac{\frac{p}{m_0}}{\sqrt{1+\frac{p^2}{{m_0}^2\,c^2}}}$ and the equality $\frac{dp}{dt} = v\,\frac{dp}{dl}$ equation (9) can be re-written and integrated between l_0 and l .Then the following equalities are obtained:

$$\int_{l_0}^{l}\frac{\frac{p}{m_0}\,\frac{dp}{dl'}}{\sqrt{1+\frac{p^2}{{m_0}^2\,c^2}}}\,dl' = \int_{l_0}^{l}\vec{f}(l')\cdot\vec{t}_\Gamma(l')\,d\,l' \tag{13 a}$$

$$\int_{l_0}^{l} \frac{\frac{p}{m_0}\frac{dp}{dl'}}{\sqrt{1+\frac{p^2}{m_0{}^2\,c^2}}}\, dl' = m_0\, c^2 \left(\sqrt{1+\frac{p^2(l)}{m_0{}^2\,c^2}} - \sqrt{1+\frac{p^2(l_0)}{m_0{}^2\,c^2}}\right) \tag{13 b}$$

From (13 a) and (13 b) we obtain $p\big(t(l)\big)$ and inserting it in (12) we derive:

$$\int_{t_0}^{+\infty} \sqrt{1-\frac{v^2(t)}{c^2}}\, dt = \frac{1}{c}\int_{l_0}^{+\infty} \frac{dl}{\sqrt{\left[\left(\sqrt{1+\frac{p^2(l_0)}{m_0{}^2\,c^2}} + \frac{\int_{l_0}^{l}\vec{f}(l')\cdot\vec{t}_\Gamma(l')\,dl'}{m_0\,c^2}\right)^2 - 1\right]}} \tag{14}$$

In our previous work about the slowing down of accelerated clocks, the convergence of the integral that gives the proper time as a function of inertial time was studied assuming that the clock moved parallel to the x-axis of the inertial system. Formulae (11) and (14) allow us to generalize the analysis to 3-D.

If $\left|\int_{t_0}^{t}\vec{f}(t')\cdot\vec{t}_\Gamma(t')\,dt'\right|$ in equation (11) or $\left|\int_{l_0}^{l}\vec{f}(l')\cdot\vec{t}_\Gamma(l')\,dl'\right|$ in equation (14) are bounded, the integral $\int_{t_0}^{+\infty}\sqrt{1-\frac{v^2(t)}{c^2}}\,dt$ diverges and we have no problem.

A necessary condition for the convergence of the proper time integral is to have unbounded either $\left|\int_{t_0}^{t}\vec{f}(t')\cdot\vec{t}_\Gamma(t')\,d\,t'\right|$ as a function of the inertial time in equation (11) or $\left|\int_{l_0}^{l}\vec{f}(l')\cdot\vec{t}_\Gamma(l')\,dl'\right|$ as a function of the arc length of the clock's path in equation (14), when $t\to+\infty$ or $l\to+\infty$ respectively. In this case $\int_{t_0}^{+\infty}\sqrt{1-\frac{v^2(t)}{c^2}}\,dt$ converges (diverges) if and only if either $\int_{t_0}^{+\infty}\frac{1}{\left|\int_{t_0}^{t}\vec{f}(t')\cdot\vec{t}_\Gamma(t')\,dt'\right|}\,dt$ or $\int_{l_0}^{+\infty}\frac{dl}{\left|\int_{l_0}^{l}\vec{f}(l')\cdot\vec{t}_\Gamma(l')\,dl'\right|}$ converges (diverges). [5]

To simplify the analysis, from now on we assume that both $\vec{f}(t)\cdot\vec{t}_\Gamma(t)$ and $\vec{f}(l)\cdot\vec{t}_\Gamma(l)$ are always positive.

Let us suppose first that the function of inertial time giving the tangential component of the relativistic 3-force along the path of the accelerated clock verifies $\vec{f}(t)\cdot\vec{t}_\Gamma(t)\approx\ln t$, asymptotically when $t\to+\infty$

[5] When the relativistic 3-force derives from a potential field $\vec{f}(\vec{r})=-\nabla U(\vec{r})$ and $\vec{f}(l)\cdot\vec{t}_\Gamma(l)=-\frac{dU}{dl}$ so $\int_{l_0}^{l}\vec{f}(l')\cdot\vec{t}_\Gamma(l')\,dl'=U(l_0)-U(l)$ and the convergence or divergence of the proper time integral reduces to the convergence or divergence of $\int_{l_0}^{+\infty}\frac{dl}{|U(l)|}$

Then, asymptotically $\int_{t_0}^{t} \vec{f}(t') \cdot \vec{t}_\Gamma(t')\, dt' \approx t \cdot \ln t$ and $\int_{t_0}^{+\infty} \frac{1}{\int_{t_0}^{t} \vec{f}(t') \cdot \vec{t}_\Gamma(t')\, dt'}\, dt$ behaves as $\int_{t_0}^{+\infty} \frac{dt}{t\, \ln t}$ which is divergent.

So $\int_{t_0}^{+\infty} \sqrt{1 - \frac{v^2(t)}{c^2}}\, dt$ diverges, and the proper time of the accelerated clock tends to infinity with the inertial time.

Let us suppose now that, asymptotically ($t \to +\infty$), $\vec{f}(t) \cdot \vec{t}_\Gamma(t) \approx t^q$ with $q > 0$. Then $\int_{t_0}^{+\infty} \frac{1}{\int_{t_0}^{t} \vec{f}(t') \cdot \vec{t}_\Gamma(t')\, dt'}\, dt$ behaves as $\int_{t_0}^{+\infty} \frac{dt}{t^{q+1}}$ which is convergent. As consequence $\int_{t_0}^{+\infty} \sqrt{1 - \frac{v^2(t)}{c^2}}\, dt$ converges and the proper time of the accelerated clock tends to a finite value meanwhile the inertial time tends to infinity.

Similarly, if asymptotically (for $l \to +\infty$) $\vec{f}(l) \cdot \vec{t}_\Gamma(l) \approx \ln l$, then the proper time of the accelerated clock tends to infinity with the inertial time. But if asymptotically (for $l \to +\infty$) $\vec{f}(l) \cdot \vec{t}_\Gamma(l) \approx l^q$ ($q > 0$) the proper time of the accelerated clock tends to a finite value meanwhile the inertial time tends to infinity.

In all these cases the tangential component of the 3-force along the path is a growing function of inertial time or path length. When this component grows as the logarithm of inertial time or path length, as time or path length tends to infinity, the integral that gives the proper time of the accelerated clock diverges. When the tangential component grows as a positive power of inertial time or path length, the integral that gives the proper time converges[6].

So nothing special (besides the rate of growth towards infinity) seems to characterize the behavior of the tangent component of fields of 3-force along the path of the accelerated clock, in relation with the convergence or divergence of the integral that gives proper time as function of inertial time. Now let us change our point of view to see if we can identify a physically meaningful difference.

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[6] If the relativistic force field derives from a potential field, such that $|U(l)|$ grows when $l \to +\infty$ along the path of the accelerated clock at least as l^p with $p > 1$, then the improper integral $\int_{l_0}^{+\infty} \frac{dl}{|U(l)|}$ converges, and as consequence the proper time integral $\int_{t_0}^{+\infty} \sqrt{1 - \frac{v^2(t)}{c^2}}\, dt$ is convergent. However, if asymptotically when $l \to +\infty$ along the path of the accelerated clock, $|U(l)| = |l\, \ln l|$ with $p \le 1$, then $\int_{l_0}^{+\infty} \frac{dl}{|U(l)|}$ diverges and $\int_{t_0}^{+\infty} \sqrt{1 - \frac{v^2(t)}{c^2}}\, dt$ is divergent. In both cases the potential tends to minus infinity along the path of the accelerated clock, but with different speeds. Of course, these remarks can't be applied to a bounded motion like an accelerated circular motion relative to the inertial frame. In this last case $U(l)$ will be a periodic function of path length.

(C) The clock hypothesis and the behavior of proper acceleration

As *there is a smooth one to one correspondence between inertial time* t *and proper time* τ , let us make a change of time variable in equation (9) putting all the dependent variables defined with respect to the inertial frame S as functions of the proper time of the accelerated clock. Then: $\frac{dp}{dt} = \frac{dp}{d\tau}\frac{d\tau}{dt} = \vec{f} \cdot \vec{t}_\Gamma$

By a straightforward calculation it is possible to show that $\frac{dp}{d\tau}\frac{d\tau}{dt} = \frac{m_0}{\left(1-\frac{v^2(\tau)}{c^2}\right)}\frac{d}{d\tau}v(\tau)$

So the relativistic extension of newton's second law can be written thus:

$$\frac{1}{\left(1-\frac{v^2(\tau)}{c^2}\right)}\frac{d}{d\tau}v(\tau) = \frac{\vec{f}(\tau)\cdot\vec{t}_\Gamma(\tau)}{m_0} \tag{15}$$

Integrating this last equation we obtain the following speed formula:

$$v(\tau) = c\ \tanh\left(\psi_\Gamma(\tau) + \tanh^{-1}\frac{v(\tau_0)}{c}\right) \tag{16}$$

In equation (16), by definition: $$\psi_\Gamma(\tau) = \frac{1}{m_0\ c}\int_{\tau_0}^{\tau}\vec{f}(\tau')\cdot\vec{t}_\Gamma(\tau')\ d\tau' \tag{17}$$

Now we consider the family $\{S^0(t)\}$ of momentarily commoving inertial frames. The frame $S^0(t)$ moves with the same velocity as the accelerated clock (at instant t of inertial time relative to S), so the clock is momentarily at rest in this frame. But it is accelerated relative to $S^0(t)$ and its acceleration could be measured by an accelerometer fixed to the clock, giving the proper acceleration $\vec{a}^0$ of the clock (Hobson et al, 2006; Rindler, 2006; Gourgoulhon, 2013).

Between the relativistic 3-force $\vec{f}$ acting on the clock, seen from S , and the relativistic 3-force $\vec{f}^0$ (proper 3-force) acting on the clock, seen from $S^0(t)$ there is the relation (Moller, 1952): $$\vec{f} = \frac{1}{\gamma_v}\vec{f}^0 + \left(1-\frac{1}{\gamma_v}\right)\frac{(\vec{f}^0\cdot\vec{v})}{v^2}\vec{v} \tag{18}$$

But $\vec{v} = v\,\vec{t}_\Gamma$ (formula 5 above) so taking the scalar product of both members of the equality (18) with $\vec{t}_\Gamma$ we obtain the remarkable relation between the tangent vector to the path, the 3-force and the 3-proper force:

$$\vec{f}(t)\cdot\vec{t}_\Gamma(t) = \vec{f}^0(\tau)\cdot\vec{t}_\Gamma(\tau) \tag{19}$$

The 3-proper force is related with the proper 3-acceleration by $\vec{f}^0 = m_0\,\vec{a}^0$ (Moller, 1952) so from this last equation and from (19) it follows:

$$\psi_\Gamma(\tau) = \frac{1}{c}\int_{\tau_0}^{\tau}\vec{a}^0(\tau')\cdot\vec{t}_\Gamma(\tau')\ d\tau' \tag{20}$$

From the equation that relates 3-forces with 3-accelerations (Moller, 1952) we obtain:

$$\vec{f} \cdot \vec{t}_\Gamma = m_0 \left(\gamma_v{}^3 \ \frac{(\vec{a} \cdot \vec{v})}{c^2} \ (\vec{v} \cdot \vec{t}_\Gamma) + \gamma_v \ (\vec{a} \cdot \vec{t}_\Gamma) \right)$$

As $\gamma_v = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$, the last equation reduces to: $\vec{f} \cdot \vec{t}_\Gamma = m_0 \ \gamma_v{}^3 (\vec{a} \cdot \vec{t}_\Gamma)$ (21)

We suppose that the tangential component $\vec{a}^0(\tau) \cdot \vec{t}_\Gamma(\tau)$ of the proper 3-acceleration is always positive. As consequence $\psi_\Gamma(\tau)$ is a strictly increasing function of τ .

Using (20), let us obtain a formula for the inertial time as a function of the proper time involving the proper acceleration. To simplify the formulae we take $v(\tau_0) = 0$.

We begin by the equation: $\frac{dt}{d\tau} = \gamma_v = \frac{1}{\sqrt{1-\frac{v^2(t)}{c^2}}} = \frac{1}{\sqrt{1-\left(\tanh(\psi_\Gamma(\tau))\right)^2}} = \cosh \psi_\Gamma(\tau)$

Integrating this equation: $t(\tau) = t(\tau_0) + \int_{\tau_0}^{\tau} \cosh \psi_\Gamma(\tau') \ d\tau'$ (22)

Here $\psi_\Gamma(\tau)$ is given, in terms of the proper acceleration of the clock, by formula (20).

The convergence of the integral $\int_{t_0}^{+\infty} \sqrt{1 - \frac{v^2(t)}{c^2}} \ dt$ is equivalent to the existence of a value τ_s of the proper time such that $\int_{\tau_0}^{\tau_s} \cosh \psi_\Gamma(\tau') \ d\tau' = +\infty$ So we have now an improper integral with a finite interval of integration and a singularity of the integrand when $\tau = \tau_s$. The integrand verifies: $\cosh \psi_\Gamma(\tau) = \frac{1}{2}\left(e^{\psi_\Gamma(\tau)} + e^{-\psi_\Gamma(\tau)}\right)$

As $\psi_\Gamma(\tau)$ is a positive and increasing function along the path of the clock, the divergence of $\int_{\tau_0}^{\tau_s} \cosh \psi_\Gamma(\tau') \ d\tau'$ reduces to the divergence of $\int_{\tau_0}^{\tau_s} e^{\psi_\Gamma(\tau')} \ d\tau'$

The integrand $e^{\psi_\Gamma(\tau)}$ has a singularity when $\tau = \tau_s$ if $\psi_\Gamma(\tau)$ presents a singularity there: $\lim_{\tau \uparrow \tau_s} \psi_\Gamma(\tau) = +\infty$ This behavior is possible if and only if $\vec{a}^0(\tau) * \vec{t}_\Gamma(\tau)$ tends to infinity when the proper time tends to τ_s from below: the tangential component of the proper 3-acceleration considered as function of proper time must have a singularity if $\int_{t_0}^{+\infty} \sqrt{1 - \frac{v^2(t)}{c^2}} \ dt$ is to converge.

If there is not such singularity for a finite proper time (because, either the increasing function $\psi_\Gamma(\tau)$ is always bounded or tends to $+\infty$ when $\tau \to +\infty$) then $\int_{\tau_0}^{\tau} \cosh \psi_\Gamma(\tau') \ d\tau'$ tends to infinity only if τ tends to infinite and then $\int_{t_0}^{+\infty} \sqrt{1 - \frac{v^2(t)}{c^2}} \ dt$ must diverge.

Only as an example of possible singularities when $\tau = \tau_s$, let us introduce a bounded positive smooth function $A(\tau)$ in the closed interval $[\tau_0, \tau_s]$ *with a positive lower bound*, as well as a positive number p such that:

$$e^{\psi_\Gamma(\tau)} = \frac{A(\tau)}{(\tau_s-\tau)^p} \tag{23}$$

From the standpoint of the convergence or divergence of improper integrals, being $A(\tau)$ bounded, $e^{\psi_\Gamma(\tau)}$ behaves as $\frac{1}{(\tau_s-\tau)^p}$ asymptotically when $\tau \uparrow \tau_s$. Then the improper integral $\int_{\tau_0}^{\tau_s} e^{\psi_\Gamma(\tau')}\, d\tau'$ is divergent if and only if the improper integral $\int_{\tau_0}^{\tau_s} \frac{1}{(\tau_s-\tau')^p}\, d\tau'$ is divergent. And this happens only if and only if $p \geq 1$.

From (20) and (23) we obtain:

$$\psi_\Gamma(\tau) = \frac{1}{c} \cdot \int_{\tau_0}^{\tau} \vec{a}^0(\tau') \cdot \vec{t}_\Gamma(\tau')\, d\tau' = \ln A(\tau) - p\, \ln(\tau_s - \tau) \tag{24}$$

Then:
$$\frac{\vec{a}^0(\tau) \cdot \vec{t}_\Gamma(\tau)}{c} = \frac{d}{d\tau}\psi_\Gamma(\tau) = \frac{1}{A(\tau)}\frac{d}{d\tau}A(\tau) + \frac{p}{\tau_s-\tau} \tag{25}$$

Having, by hypothesis, $A(\tau)$ has a positive lower bound in $[\tau_0, \tau_s]$ and being $\frac{d}{d\tau}A(\tau)$ also bounded there (if $A(\tau)$ is smooth in $[\tau_0, \tau_s]$, $\frac{d}{d\tau}A(\tau)$ must be continuous in this closed interval), $\frac{1}{A(\tau)}\frac{d}{d\tau}A(\tau)$ must remain bounded in $[\tau_0, \tau_s]$. As consequence, in this example the tangential proper 3-acceleration $\vec{a}^0(\tau) \cdot \vec{t}_\Gamma(\tau)$ behaves as $\frac{p\, c}{\tau_s-\tau}$ asymptotically when $\tau \uparrow \tau_s$.

So, seen from the viewpoint of the proper acceleration of the clock, **a physically meaningful difference appears** between the proper tangential 3-acceleration (and proper relativistic tangential 3-force histories, because $\vec{f}^0 = m_0 \cdot \vec{a}^0$) that give a finite proper time for an infinite inertial time, and those proper tangential 3-acceleration histories (and their proper relativistic tangential 3-force histories) that give an infinite proper time for an infinite inertial time: either a singularity in proper 3-acceleration (and proper 3-force) for a finite value of proper time, or its absence.

(D) Three speed histories on the path of an accelerated clock

(D.1) Consider first the speed history:

$$v(t) = c\sqrt{1 - k_s^2\, e^{-\frac{k_s}{\tau_s}t}} \qquad (t \geq 0) \tag{26}$$

Both τ_s and k_s are positive, being $k_s < 1$ in order to have $\frac{d}{dt}v(0)$ finite. Here $t_0 = 0$ and $\tau(t_0) = 0$ In this case the following function gives proper time as a function of inertial time:

$$\tau(t) = \int_0^t \sqrt{1 - \frac{v^2(t')}{c^2}}\, dt' = \tau_s\left(1 - e^{-\frac{k_s}{\tau_s}t}\right) \quad (t \geq 0) \tag{27}$$

The inverse function is: $t(\tau) = -\frac{\tau_s}{k_s} \ln\left(1 - \frac{\tau}{\tau_s}\right) \qquad (0 \le \tau < \tau_s) \qquad (28)$

From (26) and (28) we derive the speed relative to the inertial frame as a function of proper time:

$$v(\tau) = c\sqrt{1 - k_s^2\left(1 - \frac{\tau}{\tau_s}\right)^2} \qquad (0 \le \tau < \tau_s) \qquad (29)$$

Substituting (29) in (15) and taking into account (19) it follows:

$$\vec{a}^0(\tau) \cdot \vec{t}_\Gamma(\tau) = \frac{\vec{f}^0(\tau) \cdot \vec{t}_\Gamma(\tau)}{m_0} = \frac{c}{\left(\sqrt{1-k_s^2\left(1-\frac{\tau}{\tau_s}\right)^2}\right)} \frac{1}{(\tau_s - \tau)} \qquad (0 \le \tau < \tau_s) \qquad (30)$$

The tangential component of the proper 3-acceleration has a pole of the first order for $\tau = \tau_s$, because $\frac{c}{\left(\sqrt{1-k_s^2\left(1-\frac{\tau}{\tau_s}\right)^2}\right)}$ is a smooth function of the proper time.

In inertial time, from equations (9) and (26) we obtain for the tangential component of the relativistic 3-force: $\frac{\vec{f}(t) \cdot \vec{t}_\Gamma(t)}{m_0} = \frac{1}{\tau_s} \frac{e^{\frac{2k_s}{\tau_s}t}}{\sqrt{e^{\frac{2k_s}{\tau_s}t} - k_s^2}} \qquad (t \ge 0) \qquad (31)$

Asymptotically, for $t \to +\infty$: $\frac{\vec{f}(t) \cdot \vec{t}_\Gamma(t)}{m_0} \approx \frac{1}{\tau_s} e^{\frac{k_s}{\tau_s}t} \qquad (32)$

(D.2) Consider a second speed history on the path of the accelerated clock:

$$v(t) = c\sqrt{1 - \left(\frac{1}{1+\frac{t}{t_0}}\right)^{2p}} \qquad (t \ge 0\ ) \qquad (\ p > 1\ ) \qquad (33)$$

In this case $\sqrt{1 - \frac{v^2(t)}{c^2}} = \frac{1}{\left(1+\frac{t}{t_0}\right)^p}$ Proper time as function of inertial time is given by:

$$\tau(t) = \int_0^t \sqrt{1 - \frac{v^2(t')}{c^2}}\ dt' = \frac{t_0}{p-1}\left[1 - \frac{1}{\left(1+\frac{t}{t_0}\right)^{p-1}}\right] \qquad (34)$$

Now: $\lim_{t\to+\infty} \tau(t) = \tau_s = \frac{t_0}{p-1} \qquad (\ p > 1\ ) \qquad (35)$

The inverse function is: $t(\tau) = (p-1)\,\tau_s\left[\frac{1}{\left(1-\frac{\tau}{\tau_s}\right)^{\frac{1}{p-1}}} - 1\right] \qquad (0 \le \tau < \tau_s) \qquad (36)$

From (33) and (36): $v(\tau) = c\sqrt{1 - \left(1 - \frac{\tau}{\tau_s}\right)^{\frac{2p}{p-1}}} \qquad (0 \le \tau < \tau_s) \quad (\ p > 1\ ) \quad (37)$

Substituting (37) in (15) and taking into account (19) it follows:

$$\vec{a}^0(\tau)\cdot\vec{t}_\Gamma(\tau) = \frac{\vec{f}^0(\tau)\cdot\vec{t}_\Gamma(\tau)}{m_0} = \frac{c\left(\frac{p}{p-1}\right)}{\left(\sqrt{1-\left(1-\frac{\tau}{\tau_s}\right)^{\frac{2p}{p-1}}}\right)}\,\frac{1}{(\tau_s-\tau)} \qquad (0 \le \tau < \tau_s) \qquad (35)$$

Again, also in this case the tangential component of the proper 3-acceleration has a pole of the first order for $\tau = \tau_s$. The factor that multiplies $\frac{1}{(\tau_s-\tau)}$ is a smooth function that tends to $c\left(\frac{p}{p-1}\right)$ when $\tau \uparrow \tau_s$.

In inertial time, from equations (9) and (33) we obtain for the tangential component of the relativistic 3-force:

$$\frac{\vec{f}(t)\cdot\vec{t}_\Gamma(t)}{m_0} = \frac{p\left(1+\frac{t}{t_0}\right)^{2p-1}}{\sqrt{\left(1+\frac{t}{t_0}\right)^{2p}-1}} \qquad (t \ge 0) \qquad (\ p > 1\ ) \qquad (36)$$

Asymptotically, for $t \to +\infty$:

$$\frac{\vec{f}(t)\cdot\vec{t}_\Gamma(t)}{m_0} \approx p\left(\frac{t}{t_0}\right)^{p-1} \qquad (37)$$

(D.3) Consider the following third speed history on the path Γ of the clock:

$$v(t) = \frac{\alpha\, t}{\sqrt{1+\frac{\alpha^2 t^2}{c^2}}} \qquad (t \ge 0) \qquad (38)$$

In (38) the parameter α is a constant proper acceleration tangential to the path. A non inertial observer moving with a uniform proper acceleration in a straight line is usually known as a Rindler observer (Hobson et al, 2006; Gourgoulhon, 2013).

Then:

$$\tau(t) = \int_0^t \sqrt{1-\frac{v^2(t')}{c^2}}\ dt' = \frac{c}{\alpha}\ \ln\left(\frac{\alpha\, t}{c} + \sqrt{1+\frac{\alpha^2 t^2}{c^2}}\right) \qquad (t \ge 0) \qquad (39)$$

From (39) if follows, asymptotically for $t \to +\infty$:

$$\tau(t) \approx \frac{c}{\alpha}\ \ln\left(\frac{2\,\alpha\, t}{c}\right) \qquad (40)$$

So $\lim_{t\to+\infty}\tau(t) = +\infty$ An infinite proper time corresponds to an infinite inertial time for this speed history, unlike what happened in the two previous speed stories.

The inverse function is (Hobson et al, 2006):

$$t(\tau) = \frac{c}{\alpha}\ \sinh\left(\frac{\alpha\ \tau}{c}\right) \qquad (\tau \ge 0) \qquad (41)$$

The speed relative to the inertial frame, but as a function of proper time is given by the known formula:

$$v(\tau) = c\ \tanh\left(\frac{\alpha\ \tau}{c}\right) \qquad (\tau \ge 0) \qquad (42)$$

The tangential acceleration verifies: $\vec{a}^0(\tau)\cdot\vec{t}_\Gamma(\tau) = \alpha$ The tangential component of the 3-force relative to the inertial system is constant: $\vec{f}(t)\cdot\vec{t}_\Gamma(t) = m_0\,\alpha$

(E) Discussion and conclusions

All the previous analysis about the relationship between proper time and inertial time is based on the validity of the clock hypothesis and the concept of ideal clock: the rate of an accelerated ideal clock is equal to the rate of a commoving non-accelerated clock, regardless of the history of clock speed with respect to a reference inertial system. Thus, no limitations are introduced on the permissible proper accelerations[7].

In absence of gravitational effects, and in the framework of special relativity, a clock fixed to an inertial system is isolated from external actions and if properly constructed could be considered as an ideal clock. After accelerating, one would expect it to behave as much closer to an ideal clock the smaller the ratio between the magnitude of external forces associated with acceleration and the magnitude of the internal cohesion forces associated with the integrity of the material structure of clock. An ideal clock would be one totally unaffected by accelerations.

The clock hypothesis was confirmed for lifetimes of positive and negative muons stored in a circular orbit with transversal accelerations, and for lifetimes of Sigma baryons with longitudinal accelerations (Bailey et al, 1977; Bailey et al, 1979; Roos et al, 1980). The observed agreement between the lifetimes of the accelerated particles with that of the same kind of particles, with the same energy, moving inertially, confirms the clock hypothesis for accelerations of almost 10^{19} m/s^2 in the first case and almost 10^{16} m/s^2 in the second case.

Now, let us discuss some consequences of the results obtained in this work, admitting the clock hypothesis.

In the study of the convergence of the integral $\int_{t_0}^{+\infty} \sqrt{1-\frac{v^2(t)}{c^2}}\, dt$ the path Γ of the accelerated clock was considered as known.

The results were obtained in terms of the tangential components of the fields and accelerations relative to the path. Equation (3) relating the length of the path with the speed history can be rewritten, taking $t_0 = 0$ and $l(0) > 0$:

$$l(t) = l(0) + \int_0^t v(t')\, dt' \qquad (t \geq 0) \tag{43}$$

[7] However, some recent work, either of reinterpretation of known experiments (Friedman, 2010; Potzel, 2014) or theoretical developments (Schuller, 2002; Papini, 2003), suggests the possible existence of maximum accelerations. If this is correct, in the framework of special relativity the clock hypothesis would be only an approximation to a more general relation.

If $\lim_{t\to+\infty} v(t) = c$ we have $\lim_{t\to+\infty} \frac{1}{t}\int_0^t v(t')\,dt' = c$ so $\lim_{t\to+\infty} \frac{l(t)}{t} = c$ and asymptotically for $t \to +\infty$: [8]

$$l(t) \approx c\,t \tag{44}$$

Besides, the function $l(t) - c\,t = \delta(t)$ must be positive and monotonously decreasing towards 0. [9] So the function $l_{asymp} = c\,t$ is the asymptote to $l(t)$ for $t \to +\infty$ and is always less than $l(t)$. This behavior of the 3-D arc length as function of inertial time leads us to the event horizons in the theory of special relativity (Hobson el al, 2006; Rindler, 2006; Gourgoulhon, 2013).

These event horizons appear for accelerated observers in the framework of Minkowski space-time. As this space-time is not modified by matter, the characteristics of the event horizons in special relativity depend on the particular kinetic behavior of the accelerated observer. To study in depth the consequences of this behavior the path Γ and the velocity history $\vec{v}(t)$ must be considered in detail relative to an inertial system S [10].

Moreover, we have seen that the integral that gives the proper time as a function of the inertial time could converge for suitable speed histories of the accelerated clock on its path. Then, while the inertial time tends to infinity, upper bounds τ_s for proper time could appear, such that for the inertial observer the accelerated clock would take infinite time to approach to the "proper age" τ_s . Admitting that the proper time is not bounded, when $\tau > \tau_s$ the evolution of the accelerated clock is beyond the possibilities of description accessible to the inertial observer: his time has run out.

Births and deaths, explosions and implosions occurring next to the accelerated clock for proper times greater than τ_s wouldn't be facts on which *every observer in the universe* (inertial or non-inertial) could agree: the unity of space-time will have been broken.

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[8] Integrating by parts: $\int_0^t v(t')\,dt' = t\,v(t) - \int_0^t t'\,\frac{d}{dt'}v(t')\,dt'$ The speed acceleration, for any increasing speed history (necessarily bounded by the speed of light in vacuum), verifies: $\lim_{t\to+\infty} \frac{d}{dt}v(t) = 0$ So: $\lim_{t\to+\infty} \frac{1}{t}\int_0^t v(t')\,dt' = \lim_{t\to+\infty} v(t) - \lim_{t\to+\infty} \frac{\int_0^t t'\,\frac{d}{dt'}v(t')\,dt'}{t}$ For every $\epsilon > 0$ there is an instant of inertial time $T(\epsilon)$ such that for every $t > T(\epsilon)$, $\left|\frac{dv(t)}{dt}\right| < \epsilon$
As consequence, for $t > T(\epsilon)$ $\left|\frac{1}{t}\int_0^t t'\,\frac{d}{dt'}v(t')\,dt'\right| < \left|\frac{1}{t}\int_0^T t'\,\frac{d}{dt'}v(t')\,dt'\right| + \epsilon\left(1 - \frac{T}{t}\right)$ so $\lim_{t\to+\infty}\left|\frac{\int_0^t t'\frac{d}{dt'}v(t')\,dt'}{t}\right| < \epsilon$ Being ϵ arbitrary, it follows: $\lim_{t\to+\infty} \frac{\int_0^t t'\,\frac{d}{dt'}v(t')\,dt'}{t} = 0$

[9] We have: $\delta(0) = l(0) > 0$ $\frac{d}{dt}\delta(t) = v(t) - c < 0$ and $\frac{d^2}{dt^2}\delta(t) = \frac{d}{dt}v(t) < 0$ ($t \geq 0$). But we know that $\lim_{t\to+\infty} \delta(t) = 0$ If $\delta(t_*) = 0$ for some $t_* > 0$ the second derivative of $\delta(t)$ must change its sign from negative to positive if $\delta(t)$ ends up tending to zero which in this case is not possible.

[10] However, let us suppose that the space points corresponding to Γ are part of a straight line £ that contains the origin of the 3-D space (relative to S). The arc length of this line can be chosen such that the origin is given by $l = 0$. If a light pulse is emitted when $t = 0$ in the direction of £ towards the accelerated clock, it will never reach the clock and an observer moving with the clock will remain unaware of the emission event.

This behavior reminds us what people think that happens during the fall of a massive particle towards the event horizon of a black hole. But the event horizon of a black hole is a structure intrinsic to space-time in the framework of general relativity. Here we have, as already said before, a structure depending on the particular kinetic behavior of the accelerated observer.

To conclude:

(1) Nothing special seems to characterize the behavior of the tangent component of fields of 3-force, along the path of the accelerated clock and relative to an inertial frame, in relation with the convergence or divergence of the integral that gives proper time as function of inertial time. The borderline tangential force histories that give either convergence or divergence, grow in both cases, the only difference appears in the rate of growth.

(2) However, seen from the viewpoint of the proper acceleration of the clock, a physically meaningful difference appears between: (a) the tangential proper acceleration histories (and tangential proper relativistic 3-force histories) that give a finite proper time for an infinite inertial time, and (b) those tangential proper 3-acceleration histories (and their proper tangential relativistic 3-force histories) that give an infinite proper time for an infinite inertial time. Either a singularity in proper 3-acceleration (and in proper 3-force) for a finite value of proper time in one case, or its absence in the other case.

(3) If we admit that proper accelerations can't have any singularity on the path of the accelerated clock, the case of finite proper time for an infinite inertial time can be excluded, and the problem that motivates this work is overcome.

(F) References

Moller, C. 1952, The Theory of Relativity, London: Oxford University Press.

Rindler, W. 1991, Introduction to Special Relativity, (2nd edition), New York: Oxford University Press.

Brown, H. 2005, Physical Relativity,Oxford: Clarendon Press.

Taylor, E. and J. Wheeler, 1992, Space-time Physics, (2nd edition) New York: Freeman.

Suárez-Antola R. and J. Ferrari, 1985, Lettere al Nuovo Cimento, 44(8), 599-600.

Hobson M.P., G. P. Efstathiou, A. N. Lasenby, 2006, General Relativity, New York: Cambridge University Press

Rindler, W. 2006, Relativity, New York: Oxford University Press.

Gourgoulhon, E. 2013, Special Relativity in General Frames, Berlin: Springer.

Bailey, H.; Borer, K.; Combley F.; Drumm H.; Krienen F.; Lange F.; Picasso E.; Ruden W. von; Farley F. J. M.; Field J.H.; Flegel W. & Hattersley P. M. (1977). "Measurements of relativistic time dilatation for positive and negative muons in a circular orbit". Nature. 268 (5618): 301–305.

J. Bailey et al., Final report on the CERN muon storage ring including the anomalous magnetic moment and the electric dipole moment of the muon, and a direct test of relativistic time dilation, Nuclear Physics B, 150, 1979.

Roos, C. E.; Marrafino, J.; Reucroft, S.; Waters, J.; Webster, M. S.; Williams, E. G. H. (1980). "σ+/- lifetimes and longitudinal acceleration". Nature. 286 (5770): 244–24.

Friedman, Y. 2010, Testing Doppler type shift for an accelerated source and determination of the universal maximum acceleration,

Potzel, W. 2014, Clock hypothesis of relativity theory, maximal acceleration, and Mösbauer spectroscopy,

Schuller, F. 2002, Born-Infeld kinematics and correction to the Thomas precession, physics Letters B **540**, 119-124.

Papini, G. Revisiting Caianiello's Maximal Acceleration, Nuovo Cim. B **117**:1325-1331,2003.